

Vector Models in \mathcal{PT} Quantum Mechanics

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Received: 30 May 2012 / Accepted: 14 Jan 2013

Abstract We present two examples of non-Hermitian Hamiltonians which consist of an unperturbed part plus a perturbation that behaves like a vector, in the framework of \mathcal{PT} quantum mechanics. The first example is a generalization of the recent work by Bender and Kalveks, wherein the E2 algebra was examined; here we consider the E3 algebra representing a particle on a sphere, and identify the critical value of coupling constant which marks the transition from real to imaginary eigenvalues. Next we analyze a model with SO(3) symmetry, and in the process extend the application of the Wigner-Eckart theorem to a non-Hermitian setting.

Keywords Non-Hermitian quantum mechanics · \mathcal{PT} quantum mechanics · Wigner-Eckhart theorem

PACS 11.30.Er · 03.65.-w · 02.30.Fn

1 Introduction

There are many situations in quantum mechanics wherein the Hamiltonian under consideration can be written as

$$H = H_0 + H_1 \tag{1}$$

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where H_0 is the unperturbed part and commutes with the generators T_i of symmetry group G :

$$[H_0, T_i] = 0 \quad (2)$$

and H_1 can be treated like a perturbation and behaves like a vector under G . We wish to generalize this situation in the context of \mathcal{PT} quantum mechanics [1,2], where the assumption that operators such as the Hamiltonian are Hermitian is relaxed, and replaced by other requirements, notably that the Hamiltonian commutes with the parity (\mathcal{P}) and time-reversal (\mathcal{T}) operators.

Interest in non-Hermitian quantum mechanics continues to grow [3], and recently a number of experiments have observed the so-called \mathcal{PT} phase transition, where the eigenvalues of a \mathcal{PT} Hamiltonian make a transition from being complex to real once a critical value of a coupling constant is reached [4], [5], [6]. Thus it is relevant to seek new \mathcal{PT} -counterparts to conventional Hamiltonians.

In this work we present two simple cases that can be described as non-Hermitian vector perturbation models where the Hamiltonian can be written as in eq (1); first we consider a particle confined to the surface of a sphere, where the Hamiltonian acts within an infinite dimensional Hilbert space, and next we consider a generic vector perturbation within a finite dimensional Hilbert space and determine the spectrum of eigenvalues using the Wigner-Eckart theorem. We find that for a range of parameters each of these models has a pure real spectrum. At critical values of the coupling the model undergoes \mathcal{PT} transitions wherein the eigenvalues become complex.

2 E3 Algebra: particle on a sphere

We begin by generalizing the analysis presented in [7]. They considered the E2 algebra which consists of elements J, u, v such that

$$[J, u] = -iv \quad [J, v] = iu \quad [u, v] = 0. \quad (3)$$

The Hamiltonian

$$h = J^2 + igu \quad (4)$$

where $J = -i\partial/\partial\theta$, $u = \sin\theta$, $v = \cos\theta$ and g is a constant, represents a 2-dimensional quantum particle restricted to radius $r = 1$.

A generalization of this is the E3 algebra and restricting the particle to the surface of a sphere ($r = 1$). This is described by the Hamiltonian

$$h = L^2 + igu_z \quad (5)$$

where L obeys

$$[L_i, L_j] = i\epsilon_{ijk}L_k \quad (6)$$

u is a vector operator whose components are given by

$$u_x = \sin \theta \cos \phi \quad (7)$$

$$u_y = \sin \theta \sin \phi \quad (8)$$

$$u_z = \cos \theta \quad (9)$$

and g is a constant. The remaining commutators are straightforward to calculate;

$$[L_i, u_j] = i\epsilon_{ijk}u_k \quad [u_i, u_j] = 0. \quad (10)$$

Following Bender and Kalveks we consider the case of even time reversal: for a wave function $\psi(\theta, \phi)$ the time reversal operator \mathcal{T} is manifested as complex conjugation:

$$\mathcal{T}\psi(\theta, \phi) = \psi^*(\theta, \phi) \quad (11)$$

hence $\mathcal{T}^2 = 1$. It is easy to verify the action of \mathcal{T} on the elements of the algebra: $\mathcal{T}L_i\mathcal{T} = -L_i$ and $\mathcal{T}u_i\mathcal{T} = u_i$. The parity operator \mathcal{P} takes ψ into the antipodal point:

$$\mathcal{P}\psi(\theta, \phi) = \psi(\pi - \theta, \phi + \pi) \quad (12)$$

so $\mathcal{P}^2 = 1$; elements transform under parity as $\mathcal{P}L_i\mathcal{P} = L_i$ and $\mathcal{P}u_i\mathcal{P} = -u_i$. Note that the Hamiltonian h in eq (5) commutes with the combined operation \mathcal{PT} but not with \mathcal{P} or \mathcal{T} individually. Now let us determine the eigenvalue spectrum of this Hamiltonian. We wish to solve

$$h\psi(\theta, \phi) = \lambda\psi(\theta, \phi) \quad (13)$$

and we try the general solution:

$$\psi(\theta, \phi) = f(\theta)e^{im\phi}. \quad (14)$$

For convenience we define $\eta = \cos\theta$; this simplifies the eigenvalue equation for f :

$$-(1 - \eta^2)\frac{\partial^2 f}{\partial \eta^2} + 2\eta\frac{\partial f}{\partial \eta} + \frac{m^2}{1 - \eta^2}f + ig\eta f = \lambda f \quad (15)$$

where m is a fixed integer. If we let

$$h_0 = -(1 - \eta^2)\frac{\partial^2 f}{\partial \eta^2} + 2\eta\frac{\partial f}{\partial \eta} + \frac{m^2}{1 - \eta^2}f \quad (16)$$

then the Hamiltonian we wish to solve is

$$h_0 f + ig\eta f = \lambda f. \quad (17)$$

We impose the boundary condition that the solution must be regular at $\eta = \pm 1$.

Let us choose basis elements

$$|l\rangle \rightarrow N_l P_{l,|m|}(\eta) \quad (18)$$

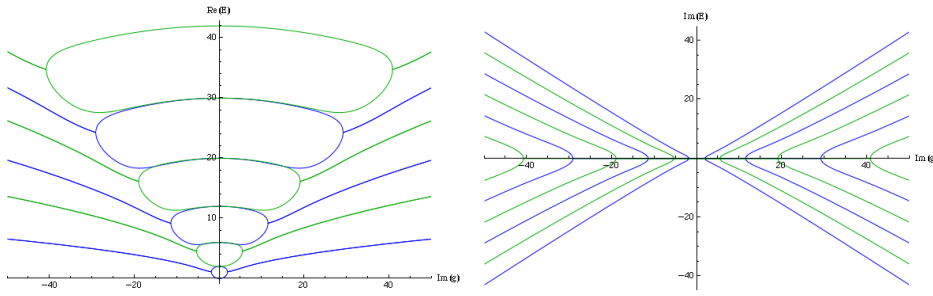


Fig. 1 Real and imaginary components of eigenvalues E for the Hamiltonian given by Eq.17. The first six eigenvalues for $m = 0$ (blue) and $m = 1$ (green) are shown. Intercepts on the E axis are given by $\ell(\ell + 1)$ for $\ell = 0$ to 6. For the case of $m = 0$, we find that the spectrum is entirely real for $0 \leq g < 1.899$ at which point there is a transition to one pair of complex conjugate eigenvalues in the spectrum. At $g = 11.45$ there is a second transition, to two pairs of complex conjugate eigenvalues. Similarly for the case of $m = 1$, we find one complex conjugate eigenvalue pair at $g = 5.41$, and two pairs at $g = 19.04$. In these computations the Hamiltonian is truncated to a 100×100 matrix; we have verified that the relevant part of the spectrum is insensitive to the truncation.

where $l = |m|, |m| + 1, \dots$, $P_{l,|m|}$ are the associated Legendre polynomials, with conventional normalization factor

$$N_l = \sqrt{\frac{(2l+1)}{2}} \sqrt{\frac{(l-|m|)!}{(l+|m|)!}}. \quad (19)$$

The $P_{l,m}$'s satisfy

$$h_0 P_{l,|m|}(\eta) = l(l+1) P_{l,|m|}(\eta) \quad (20)$$

so the matrix of h_0 in this basis is diagonal. The matrix elements of the potential term, $ig\eta$, can easily be determined from the normalization and recursion relations of the $P_{l,m}$'s. By diagonalizing the truncated Hamiltonian matrix we can numerically obtain the eigenvalues of eq (17); see Fig. (1).

3 \mathcal{PT} Vector Model in Finite-Dimensional Hilbert Space

$E3$ may also be regarded as a realization of the \mathcal{PT} vector model with symmetry group $SO(3)$ and for which the Hilbert space is infinite dimensional. Now we wish to turn our attention to realizations of the \mathcal{PT} vector model with finite dimensional Hilbert spaces. Let us write a simple, generic Hamiltonian $H = H_0 + H_I$ where

$$H_0 = L_x^2 + L_y^2 + L_z^2 \quad (21)$$

$$H_I = V_z \quad (22)$$

and V_z is the z component of a vector operator.

Our task is to obtain a matrix representation of the total Hamiltonian, solve for its eigenvalues and determine what value of the non-Hermitian perturbation cause the eigenvalues to become complex.

Naturally we choose to work with the angular momentum eigenstates $|\ell, m\rangle$; the action of H_0 on these states is well known, and we can utilize the Wigner-Eckart theorem to determine the action of $H_I = V_z$.

Note that the dimensionality of the relevant vector space depends on the angular momenta of the multiplets but clearly it is finite. Suppose we consider the two multiplets $|\ell, m\rangle$ and $|\ell + 1, m\rangle$; m takes on values from $-\ell$ to $+\ell$ in the first multiplet and from $-\ell - 1$ to $\ell + 1$ in the second multiplet, so there are $(2\ell + 1) + (2\ell + 3) = 4\ell + 4$ of these states.

The action of H_0 on these states is simply

$$L^2|\ell, m\rangle = \ell(\ell + 1)|\ell, m\rangle \quad (23)$$

$$L^2|\ell + 1, m\rangle = (\ell + 1)(\ell + 2)|\ell + 1, m\rangle. \quad (24)$$

So all that remains is to determine how V_z acts on these states; here we employ the Wigner-Eckart theorem, which we have extended to the non-Hermitian case as detailed in Appendix A. We find $\langle \ell', m' | V_z | \ell, m \rangle = 0$ unless $m = m'$. Thus we need only to determine

$$\begin{aligned} &\langle \ell, m | V_z | \ell, m \rangle, \\ &\langle \ell + 1, m | V_z | \ell + 1, m \rangle, \\ &\langle \ell, m | V_z | \ell + 1, m \rangle, \text{ and} \\ &\langle \ell + 1, m | V_z | \ell, m \rangle \end{aligned}$$

in order to completely specify V_z in this space. The first two in this list can be expressed in terms of the reduced matrix element α defined in Appendix A; in general we find

$$\begin{aligned} \langle \ell, m | V_z | \ell, m \rangle &= m\alpha_1 \\ \langle \ell + 1, m | V_z | \ell + 1, m \rangle &= m\alpha_2; \end{aligned}$$

however we also wish to enforce $\mathcal{P}V_z\mathcal{P} = -V_z$ and $\mathcal{T}V_z\mathcal{T} = -V_z$, which restricts $\alpha_1 = \alpha_2 = 0$. (Determination of \mathcal{P} and \mathcal{T} within this space follows straightforwardly from their action on the spherical harmonics $PY_{\ell m}(\theta, \phi) = (-1)^\ell Y_{\ell m}(\theta, \phi)$ and $TY_{\ell m}(\theta, \phi) = Y_{\ell m}^*(\theta, \phi) = (-1)^m Y_{\ell, -m}(\theta, \phi)$.)

For the other two types of matrix elements, $\langle \ell, m | V_z | \ell + 1, m \rangle$ and $\langle \ell + 1, m | V_z | \ell, m \rangle$, we find these are proportional to other reduced matrix elements β and γ ;

$$\begin{aligned} \langle \ell + 1, m | V_z | \ell, m \rangle &= f_{\ell m}\beta, \\ \langle \ell, m | V_z | \ell + 1, m \rangle &= f_{\ell m}\gamma \end{aligned}$$

where

$$f_{\ell m} = \left[\frac{(\ell + 1)^2 - m^2}{(2\ell + 1)(2\ell + 2)} \right]^{1/2}. \quad (25)$$

Note that $f_{\ell m}$ is even in m . When we enforce $PV_zP = -V_z$ and $TV_zT = -V_z$, we find this requires β and γ to be pure imaginary, so we define $\beta = ib$, $\gamma = ic$ for some real numbers b, c . Note that in determining these matrix elements we do not assume V is Hermitian; we rely only on the commutators of V with the angular momentum operators. (See Appendix A for details.)

Now let us write down the matrix corresponding to the Hamiltonian $H = H_0 + H_I$. Consider the two-dimensional subspace spanned by the states $|\ell, m\rangle$ and $|\ell + 1, m\rangle$ for a fixed value of m that lies in the range $-\ell, \dots, \ell$. Within this subspace

$$H_0 = \begin{pmatrix} \ell(\ell+1) & 0 \\ 0 & (\ell+1)(\ell+2) \end{pmatrix} \quad (26)$$

and

$$V_z = \begin{pmatrix} 0 & -icf_{\ell m} \\ -ibf_{\ell m} & 0 \end{pmatrix}. \quad (27)$$

In addition consider the two dimensional subspace spanned by the states $|\ell + 1, \ell + 1\rangle$ and $|\ell + 1, -\ell - 1\rangle$. These states are not coupled by the perturbation V_z to any other state and hence $V_z = 0$ within this subspace. On the other hand the unperturbed Hamiltonian in this subspace is given by

$$H_0 = \begin{pmatrix} (\ell+1)(\ell+2) & 0 \\ 0 & (\ell+1)(\ell+2) \end{pmatrix}. \quad (28)$$

It is convenient to define

$$h_{\ell+1} = \begin{pmatrix} (\ell+1)(\ell+2) & 0 \\ 0 & (\ell+1)(\ell+2) \end{pmatrix} \quad (29)$$

and

$$h_m = \begin{pmatrix} \ell(\ell+1) & -icf_{\ell m} \\ -ibf_{\ell m} & (\ell+1)(\ell+2) \end{pmatrix} \quad (30)$$

where $m = -\ell, \dots, \ell$. The Hamiltonian can now be written as a block-diagonal matrix

$$\begin{pmatrix} h_{\ell+1} & & & \\ & h_{\ell} & & \\ & & h_{\ell-1} & \\ & & & \ddots \\ & & & & h_{-\ell} \end{pmatrix}. \quad (31)$$

The individual 2×2 matrices that constitute the Hamiltonian are simple enough that we can obtain analytic expressions for the eigenvalues. The eigenvalues of $h_{\ell+1}$ are two-fold degenerate and are simply $(\ell+1)(\ell+2)$. The eigenvalues of h_m are

$$\lambda_{\ell m}^{\pm} = (\ell+1)^2 \pm \sqrt{(\ell+1)^2 - bcf_{\ell m}^2} \quad (32)$$

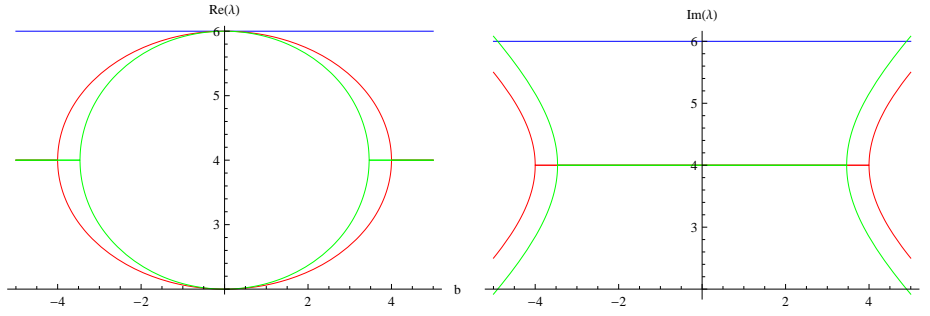


Fig. 2 Real and imaginary parts of the eigenvalues $\lambda_{1,m}$ assuming $b = c$. The blue line at $\lambda = 6$ corresponds to the 2×2 matrix denoted $h_{\ell+1}$ in the text, with eigenvalues $(\ell+1)(\ell+2)$. The other eigenvalues are m -dependent and correspond to the 2×2 matrices denoted h_m in the text, with eigenvalues given by eq(32). As noted $\lambda_{\ell,m} = \lambda_{\ell,-m}$, so there are only two distinct m -dependent curves for $\ell = 1$. In each figure, $|m| = 1$ is plotted in red and $m = 0$ is plotted in green. Note that the transition to complex eigenvalues occurs at $b = 4$ for $\lambda_{1,1}^{\pm}$ and $b = \sqrt{12} \approx 3.47$ for $\lambda_{1,0}^{\pm}$.

Note that $\lambda_{\ell,m} = \lambda_{\ell,-m}$ so for all $m \neq 0$ the eigenvalues of h_m are also two-fold degenerate. Clearly, the eigenvalues are real provided

$$bc < \frac{(\ell+1)^2}{f_{lm}^2}. \quad (33)$$

We can make the following observations about the behavior of the eigenvalues. Once \mathcal{PT} symmetry is broken, λ^+ and λ^- form a complex conjugate pair. Since $f_{\ell m}$ has its maximum value for $m = 0$, λ^{\pm} becomes complex for $m = 0$ first. Similarly, $f_{\ell m}$ is minimal for $|m| = \ell$, so λ^{\pm} so these are the last eigenvalues to go complex. For example we consider the case $\ell = 1$ and choose $b = c$ for simplicity. We plot the eigenvalues in Figure 2.

It is worth noting that in the Hermitian case $b = -c$. Hence the condition in eq (33) that ensures the eigenvalues are real is always met.

4 Conclusion

We conclude by noting two natural generalizations of our results that deserve further investigation. First the model of a particle on an ordinary 2-sphere considered in section II may be generalized to a particle on a sphere in n dimensions. The \mathcal{PT} transition for this model may be amenable to analytic study in the large n limit and may shed some light on \mathcal{PT} symmetric non-linear sigma models of which it would represent a $0 + 1$ dimensional case [8]. Second the vector model constructed in section III may be easily generalized from the symmetry group $SO(3)$ to any Lie group and therefore represents only one member of a large class of such models.

Appendix A: Wigner-Eckart Theorem

Suppose we have an angular momentum operator \mathbf{L} and a vector operator \mathbf{V} satisfying the commutation relations

$$[L_i, V_j] = i\epsilon_{ijk}V_k. \quad (34)$$

Let $|\ell, m\rangle$ denote an angular momentum multiplet of total angular momentum ℓ and z -component m . Then according to the Wigner-Eckart theorem the matrix elements of V_z and $V_{\pm} = V_x \pm iV_y$ between multiplet states are determined by the commutation relations eq (34). In the usual Wigner-Eckart theorem the Cartesian components of the operator \mathbf{V} are assumed to be hermitian. Here we present a non-Hermitian generalization of the theorem.

Following the usual arguments we find the selection rules

$$\langle \ell', m' | V_z | \ell, m \rangle = 0 \quad \text{unless } m' = m \quad (35)$$

$$\langle \ell', m' | V_+ | \ell, m \rangle = 0 \quad \text{unless } m' = m + 1 \quad (36)$$

$$\langle \ell', m' | V_- | \ell, m \rangle = 0 \quad \text{unless } m' = m - 1. \quad (37)$$

Furthermore the matrix elements vanish unless $\ell' = \ell - 1$ or $\ell' = \ell$ or $\ell' = \ell + 1$.

Consider the case $\ell' = \ell$. Generalization of the usual arguments shows that

$$\begin{aligned} \langle \ell, m + 1 | V_+ | \ell, m \rangle &= A(\ell - m)^{1/2}(\ell + m + 1)^{1/2} & m = -\ell, \dots, \ell - 1 \\ \langle \ell, m | V_z | \ell, m \rangle &= Am & m = -\ell, \dots, \ell \\ \langle \ell, m - 1 | V_- | \ell, m \rangle &= A(\ell - m + 1)^{1/2}(\ell + m)^{1/2} & m = -\ell + 1, \dots, \ell \end{aligned} \quad (38)$$

where the proportionality constant A is a complex number called the “reduced matrix element”. Note that for \mathbf{V} hermitian, A would have to be real, but there is no such restriction in the non-hermitian case.

Similarly in the case $\ell' = \ell + 1$ we find

$$\begin{aligned} \langle \ell + 1, m + 1 | V_+ | \ell, m \rangle &= B \left[\frac{(\ell + m + 2)(\ell + m + 1)}{(2\ell + 2)(2\ell + 1)} \right]^{1/2} \\ \langle \ell + 1, m | V_z | \ell, m \rangle &= -B \left[\frac{(\ell - m + 1)(\ell + m + 1)}{(2\ell + 2)(2\ell + 1)} \right]^{1/2} \\ \langle \ell + 1, m - 1 | V_- | \ell, m \rangle &= -B \left[\frac{(\ell - m + 1)(\ell - m + 2)}{(2\ell + 2)(2\ell + 1)} \right]^{1/2}, \end{aligned} \quad (39)$$

where $m = -\ell, \dots, \ell$ and B is another complex reduced matrix element.

Finally in the case that $\ell' = \ell - 1$ we find

$$\begin{aligned}\langle \ell - 1, m + 1 | V_+ | \ell, m \rangle &= -C \left[\frac{(\ell - m - 1)(\ell - m)}{(2\ell)(2\ell - 1)} \right]^{1/2} \\ \langle \ell - 1, m | V_z | \ell, m \rangle &= -C \left[\frac{(\ell - m)(\ell + m)}{(2\ell)(2\ell - 1)} \right]^{1/2} \\ \langle \ell - 1, m - 1 | V_- | \ell, m \rangle &= C \left[\frac{(\ell + m)(\ell + m - 1)}{(2\ell)(2\ell - 1)} \right]^{1/2},\end{aligned}\quad (40)$$

where C is a complex reduced matrix element and $m = -\ell, \dots, \ell - 2$ in the first line of eq (40), $m = -\ell + 1, \dots, \ell - 1$ in the second line of eq (40), and $m = -\ell + 2, \dots, \ell$ in the last line of eq (40).

In the hermitian case the reduced matrix elements satisfy $B = C^*$ but in the non-hermitian case there is no such restriction on the complex elements B and C .

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